Practical Synchronization of Heterogeneous Multi-agent System Using Adaptive Law for Coupling Gains

Seungjoon Lee, Hyeonjun Yun and Hyungbo Shim

Abstract— **Practical synchronization of the heterogeneous multi-agent system is studied in this paper. In particular, we propose an adaptive law to adjust the coupling gains to achieve practical synchronization in a fully distributed manner without the need of any global information such as the total number of agents in the network or the algebraic connectivity of the communication topology. In addition, a distributed protocol is proposed such that the performance of practical synchronization becomes independent of any global information as well as the addition of new agent.**

I. INTRODUCTION

In this paper, we study synchronization of the multi-agent system with N agents given by

$$\dot{x}_i = f_i(t, x_i) + u_i, \quad i \in \mathcal{N} \tag{1}$$

where $\mathcal{N} := \{1, \ldots, N\}$ and $x_i, u_i \in \mathbb{R}^n$ are states and input, respectively. The vector field $f_i(t, x_i) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ represents the nonlinear dynamics of each agent. We suppose that individual agent is interconnected via diffusive-type coupling given as

$$u_i = k_i(t) \sum_{j \in \mathcal{N}_i} (x_j - x_i), \quad \forall i \in \mathcal{N} \tag{2}$$

where $k_i(t) \in \mathbb{R}$ is the (time-varying) coupling gain and $\mathcal{N}_i$ is the set of neighbors of node i for a connected, undirected graph. For the system (1)-(2), practical synchronization of the agents will be studied which is defined as below.

Definition 1: The system (1)-(2) achieves practical synchronization if, for given $\epsilon > 0$,

$$\limsup_{t \to \infty} |x_i(t) - x_j(t)| \leq \epsilon, \quad \forall i, j \in \mathcal{N}.$$

Moreover, we say that the system achieves asymptotic synchronization if the above inequality holds with $\epsilon = 0$.

Many studies have been done for practical synchronization for the system of the form (1) with *static* and *identical* coupling gain, i.e., $k_i(t) = k$. For example, it has been shown that the system achieves practical synchronization with sufficiently high coupling gain for a class of system satisfying QUAD property in [1], Lipschitz-type bound in [2], and the ultimate boundedness of the interconnected system in [3].

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (Ministry of Science and ICT) (No. NRF-2017R1E1A1A03070342).

S. Lee and H. Shim are with ASRI, Department of Electrical and Computer Engineering, Seoul National University, Seoul, Korea. seungjoon.lee@cdsl.kr, hshim@snu.ac.kr

H. Yun is with Test and Package Technology Group, Mechatronics R&D Center, Samsung Electronics, Hwaseong, Korea. hjyun.w@gmail.com

However, for the works mentioned above, the ultimate bound of the synchronization error (i.e., ϵ in Definition 1) depends on the coupling gain k as well as global information of the overall system such as the total number of agents in the network N, algebraic connectivity of the communication topology, or the vector fields of agents. Therefore, such global information must be known beforehand in order to design an appropriate coupling gain achieving the desired performance. To tackle this problem, adaptive laws have been studied to increase the coupling gains. For example, an edge-based adaptive strategy was proposed in [5], [6] and a node-based adaptive law was designed in [7], [8] to achieve fully distributed control without the need of any global information. However, these works focused on asymptotic synchronization of homogeneous multi-agent system.

In this paper, we propose an adaptive law to adjust gains based on local information to achieve practical synchronization of heterogeneous agents. An advantage of the proposed solution works such as [1] and [3] is that it can be implemented and designed in a fully distributed manner. In addition, it has been reported that the ultimate bound of the practical synchronization depends on global information and may increase with larger N [2], [4]. In particular, [2] showed that the coupling gain k should be increased with larger N to maintain the ultimate bound of the practical synchronization. Therefore, a distributed protocol will be proposed such that the performance of practical synchronization is guaranteed regardless of any global information including the total number of agents in the network.

This paper is organized as follows. In Section II, we present the adaptive law to adjust the couplings gains and propose a distributed protocol to guarantee the global performance. In Section III, the proposed solution is applied to solve an optimization problem in a fully distributed manner with simulation result shown in Section IV. Finally, we conclude in Section V.

Notations: For a vector x and a set Ξ, $|x|$ denotes the Euclidean norm. In particular, $|x|_2$ denotes the Euclidean 2-norm and $|x|_\infty$ denotes the maximum norm. Also, $|x|_\Xi := \inf_{z \in \Xi} |x - z|$ and $|\Xi|$ denotes the cardinality of the set. An undirected graph is defined as $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ where $\mathcal{N} := \{1, \ldots, N\}$ is the node set and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the edge set. The adjacency matrix $A = [\alpha_{ij}] \in \mathbb{R}^{N \times N}$ is defined such that $\alpha_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $\alpha_{ij} = 0$ otherwise. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ is defined as $l_{ii} := \sum_{j \neq i} \alpha_{ij}$ and $l_{ij} := -\alpha_{ij}$ for all $i \neq j$. The incidence matrix $B = [b_{ij}] \in \mathbb{R}^{N \times E}$ with $E := |\mathcal{E}|$ is defined such that $b_{ig} := -\alpha_{ij}$ and $b_{jg} := \alpha_{ij}$ for the

g^{th} edge (i,j). By construction, we have $BB^T = L$ [8]. Denote $0 = \mu_1 \leq \ldots \leq \mu_N$ as the eigenvalues of the Laplacian matrix L for an undirected, connected graph $\mathcal{G}$. A path connecting nodes i and j with length d_{ij} is defined as a sequence of distinct nodes $\mathcal{P}_{ij} := \{m_1, m_2, \ldots, m_{d_{ij}+1}\}$ where $m_1 = i$, $m_{d_{ij}+1} = j$ and $(m_q, m_{q+1}) \in \mathcal{E}$ for all $q = 1, \ldots, d_{ij}$. The neighbors of node i is defined as $\mathcal{N}_i := \{j | (j,i) \in \mathcal{E}\}$. We denote $1_N = [1, \ldots, 1]^T \in \mathbb{R}^N$ and $\mathbb{R}_{\geq 0}$ as the set of non-negative numbers. For matrices $P_1, \ldots, P_N$, $\text{diag}(P_1, \ldots, P_N)$ denotes the block diagonal matrix with P_i's in its diagonal.

II. Main Result

A. Practical synchronization with adaptive law

To achieve practical synchronization in fully distributed manner, we propose the following adaptive law for adjusting the gains:

$$\dot{k}_i = \sum_{j \in \mathcal{N}_i} \sigma_i(e_{ij}^T e_{ij}) + \sum_{j \in \mathcal{N}_i} (k_j - k_i), \quad k_i(0) > 0 \tag{3}$$

where $e_{ij} := x_i - x_j \in \mathbb{R}^n$ is the (local) synchronization error. The deadzone function $\sigma_i(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined as

$$\sigma_i(x) = \begin{cases} x - \gamma_i, & \text{if } x > \gamma_i, \\ 0, & \text{if } x \leq \gamma_i \end{cases}$$

where $\gamma_i > 0$ is the threshold of deadzone function σ_i.

Remark 1: The adaptive law shown in (3) consists of two terms. The first term of (3) (i.e., $\sum_{j \in \mathcal{N}_i} \sigma_i(e_{ij}^T e_{ij})$) uses synchronization error to increase gains which coincides with methods proposed in [5]–[8]. However, (3) uses an additional deadzone function, and assigns gain to each agent while using error based on edges (i.e., e_{ij}) instead of error based on nodes (i.e., $\sum_{j \in \mathcal{N}_i} (x_j - x_i)$). This particular structure will play an important role in deriving the results of next section. The second term (i.e., $\sum_{j \in \mathcal{N}_i} (k_j - k_i)$) tends to synchronize the coupling gains. Finally, note that the adaptive law (3) can be designed and deployed in a fully distributed manner without the need of any global information.

The following assumption is made for the dynamics of each agent.

Assumption 1: The function f_i is continuous in each argument and bounded for all $i \in \mathcal{N}$, i.e., there exists $M_i > 0$ such that

$$|f_i(t, x_i)| \leq M_i, \quad \forall t \in \mathbb{R}_{\geq 0}, \quad \forall x_i \in \mathbb{R}^n.$$

Then, following result about the synchronization of agents and gains can be obtained.

Theorem 1: Suppose that Assumption 1 holds. Then, the solution of the system (1)-(3) satisfies

$$\limsup_{t \to \infty} e_{ij}^T(t) e_{ij}(t) \leq \gamma_i, \quad \forall j \in \mathcal{N}_i, \tag{4}$$

for all $i \in \mathcal{N}$. Moreover, there exists a constant $k^* > 0$ such that $\lim_{t \to \infty} k_i(t) = k^*$ for all $i \in \mathcal{N}$.

Proof: The proof is shown in Appendix. ∎

Result of Theorem 1 can be regarded as achieving practical synchronization in a *local* sense. Specifically, the proposed adaptive law only guarantees that an agent is close with its immediate neighbors, not with *every* agent in the network. However, recall that practical synchronization in Definition 1 was defined in a *global* sense, i.e., $\limsup_{t \to \infty} |x_i(t) - x_j(t)| \leq \epsilon$ must hold for *any* two agents i, j in the network. Therefore, we will denote ϵ as *global performance index* and γ_i as *local performance index*. In fact, local performance indices shown in (4) can be used to find a global performance index as shown in the following corollary.

Corollary 1: Suppose that (4) holds. Then, the system (1)-(3) achieves practical synchronization. In particular,

$$\limsup_{t \to \infty} |x_i(t) - x_j(t)|_2 \leq (N-1) \cdot \sqrt{\bar{\gamma}}, \quad \forall i, j \in \mathcal{N} \tag{5}$$

holds where $\bar{\gamma} := \max_{i \in \mathcal{N}} \gamma_i$.

Proof: For any two nodes i and j with a path $\mathcal{P}_{ij}$, it follows from (4) that

$$\begin{aligned} \limsup_{t \to \infty} |x_i(t) - x_j(t)|_2 &\leq \sum_{q=1}^{d_{ij}} \limsup_{t \to \infty} |e_{m_q m_{q+1}}(t)|_2 \\ &\leq \sum_{q=1}^{d_{ij}} \min(\sqrt{\gamma_{m_q}}, \sqrt{\gamma_{m_{q+1}}}) \\ &\leq (N-1)\sqrt{\bar{\gamma}}. \end{aligned} \tag{6}$$

This completes the proof. ∎

B. Distributed protocol for guaranteed global performance

It can be seen from Corollary 1 that the global performance index, $(N-1)\sqrt{\bar{\gamma}}$ of (5), increases as the number of agents in the network N increases. In fact, similar dependence was shown in other studies such as [2] and [4]. However, this implies that the global performance index may degrade as new agent joins the network. Therefore, in general, the global performance index depends on the number of agents.

To prevent the degradation of the performance, consider a multi-agent system with a desired global performance index of ϵ. In order to guarantee that the desired global performance index ϵ remains valid regardless of the addition of agents, it is observed from (6) that lowering the threshold of deadzone functions γ_i may remedy the degradation of the global performance index. Inspired by this, we propose a protocol to reduce γ_i in a distributed manner to construct a network with guaranteed global performance index as follows.

Before introducing the protocol, let $\gamma_i^{[N]}$ to denote the threshold of deadzone function σ_i with N agents in the network. Note this is used purely for the notation, and that agents do not need to know N to execute the protocol.

Suppose that the network consists of a single agent. Then let $\gamma_1^{[1]} = (2\epsilon)^2$, where $\epsilon > 0$ is the desired global performance index. Now assume that a new agent is joining the network consisting of $N-1$ agents. Let us denote this agent as agent N. Then, agents in the network execute the following protocol.

Threshold Update Protocol (TUP):

1) Agent N joins the network. Let its neighbors $\mathcal{N}_N$.
2) For all $i \in \mathcal{N}_N$, let $\hat{\gamma}_i^{[N]} := \min_{j \in \mathcal{N}_i \cup \{i\}, j \neq N} (\gamma_j^{[N-1]})$.

3) Agent N receives the value of $\hat{\gamma}_i^{[N]}$ for all $i \in \mathcal{N}_N$.
4) Agent N computes $\gamma^* := \min_{i\in\mathcal{N}_N} \hat{\gamma}_i^{[N]}$ and set $\gamma_N^{[N]} = (\sqrt{\gamma^*}/2)^2$.
5) Agent N sends γ^* to its neighbors $\mathcal{N}_N$.
6) For all $i \in \mathcal{N}_N$, let $\gamma_i^{[N]} = (\sqrt{\hat{\gamma}_i^{[N]}} - \sqrt{\gamma^*}/2)^2$.

Finally, let $\gamma_i^{[N]} = \gamma_i^{[N-1]}$ for all $i \in \{1, \ldots, N-1\}\backslash\mathcal{N}_N$.

Theorem 2: For any given $\epsilon > 0$, the system (1)-(3) following the TUP achieves practical synchronization with global performance index ϵ. In particular, it holds that

$$\limsup_{t\to\infty} |x_i(t) - x_j(t)|_2 \le \epsilon, \quad \forall i, j \in \mathcal{N}. \tag{7}$$

Proof: We proceed by induction on the number of agents in the network. If there is a single agent, set $\gamma_1^{[1]} = (2\epsilon)^2$. Next, if there are two agents, it can be checked that $\gamma_1^{[2]} = \gamma_2^{[2]} = \epsilon^2$ after executing the TUP. Therefore, we obtain $\limsup_{t\to\infty} |x_1(t) - x_2(t)|_2 \le \epsilon$ by Theorem 1.

Next, suppose there are $N-1$ agents with $N-1 \ge 2$ and assume that for all $i, j \in \{1, \ldots, N-1\}$,

$$\sum_{q=1}^{d_{ij}} \min\left(\sqrt{\gamma_{m_q}^{[N-1]}}, \sqrt{\gamma_{m_{q+1}}^{[N-1]}}\right) \le \epsilon \tag{8}$$

holds for any path $\mathcal{P}_{ij}$. Then, consider a new agent joining the network while following the TUP. We will show that

$$\sum_{q=1}^{d_{ij}} \min\left(\sqrt{\gamma_{m_q}^{[N]}}, \sqrt{\gamma_{m_{q+1}}^{[N]}}\right) \le \epsilon \tag{9}$$

holds for all $i, j \in \mathcal{N}$ and for all path $\mathcal{P}_{ij}$.

Due to (8) and TUP, it is obvious that (9) holds for all $i, j \in \{1, \ldots, N-1\}$ and for any path $\mathcal{P}_{ij}$. Meanwhile, for all $k \in \mathcal{N}_N$ and $j = N$, we have

$$\min\left(\sqrt{\gamma_k^{[N]}}, \sqrt{\gamma_N^{[N]}}\right) \le \sqrt{\gamma_N^{[N]}} = \frac{\sqrt{\gamma^*}}{2} \le \epsilon. \tag{10}$$

Finally, for any path $\mathcal{P}_{ij}$ with $i \in \{1, \ldots, N-1\}\backslash\mathcal{N}_N$, $j = N$, and for any $k \in \mathcal{N}_N$, we obtain

$$\begin{aligned}
&\sum_{q=1}^{d_{ij}} \min\left(\sqrt{\gamma_q^{[N]}}, \sqrt{\gamma_{q+1}^{[N]}}\right) \\
&\le \sum_{q=1}^{d_{ij}-1} \min\left(\sqrt{\gamma_{m_q}^{[N]}}, \sqrt{\gamma_{m_{q+1}}^{[N]}}\right) + \sqrt{\gamma_N^{[N]}} \\
&\le (\epsilon - \frac{\sqrt{\gamma^*}}{2}) + \frac{\sqrt{\gamma^*}}{2} = \epsilon
\end{aligned} \tag{11}$$

where we have used

$$\min\left(\sqrt{\gamma_{m_{d_{ij}-1}}^{[N]}}, \sqrt{\gamma_k^{[N]}}\right) \le \sqrt{\gamma_k^{[N]}} = \hat{\gamma}_k^{[N]} - \frac{\sqrt{\gamma^*}}{2},$$

(8) for $\mathcal{P}_{ik}$, and (10) to obtain (11). Therefore, (9) holds for all $i, j \in \mathcal{N}$. Thus, for any $i, j \in \mathcal{N}$, it holds that

$$\limsup_{t\to\infty} |x_i(t) - x_j(t)|_2 \le \sum_{q=1}^{d_{ij}} \min\left(\sqrt{\gamma_{m_q}^{[N]}}, \sqrt{\gamma_{m_{q+1}}^{[N]}}\right) \le \epsilon.$$

This completes the proof. ■

The TUP can be executed in a fully distributed manner. For instance, agents do not require any global information such as the total number of agents in the network, or the algebraic connectivity of the communication graph. Only the local communication between an agent and its neighbors is used to update the threshold γ_i. Nevertheless, the TUP with adaptive law (3) can guarantee that the global performance index is independent of any global information.

III. Application: Economic Dispatch Problem

In this section, we provide an example where the proposed dynamics can be applied. Consider the economic dispatch problem (EDP) [9] which is formulated as

$$\min_{x_i} \sum_{i=1}^{N} f_i(x_i) \tag{12a}$$

$$\text{s.t.} \sum_{i=1}^{N} x_i = \sum_{i}^{N} p_i^d, \tag{12b}$$

$$\underline{x}_i \le x_i \le \bar{x}_i, \quad \forall i \in \mathcal{N} \tag{12c}$$

where $x_i \in \mathbb{R}$ is power generation of node i, $f_i(x_i) = a_i x_i^2 + b_i x_i + c_i$ is the local cost function, $p_i^d \in \mathbb{R}$ is power demand at node i and $\underline{x}_i \in \mathbb{R}$ and $\bar{x}_i \in \mathbb{R}$ are minimum and maximum generation capacity of each node. Objective of the EDP (12) is to find the optimal power generation of each node while meeting the power demand and generation constraint. We assume $f_i, p_i^d, \underline{x}_i$ and $\bar{x}_i$ are only available to node i, $a_i > 0$, and that the problem (12) is feasible.

Define $x := [x_1, \ldots, x_N]^T$ and let $\lambda \in \mathbb{R}$ be the Lagrange multiplier associated with (12b). Then the corresponding Lagrangian is $\mathcal{L}(x, \lambda) := \sum_{i=1}^{N}(f_i(x_i) + \lambda(x_i - p_i^d)) := \sum_{i=1}^{N} \mathcal{L}_i(x_i, \lambda)$ where the domain of $\mathcal{L}(x,\lambda)$ is $\mathcal{D} := \{(x, \lambda) \mid x_i \in [\underline{x}_i, \bar{x}_i],\ \lambda \in \mathbb{R}\}$. The optimization problem (12) can be reformulated into its dual form [10] as

$$\max_{\lambda} \sum_{i=1}^{N} J_i(\lambda) \tag{13}$$

where $J_i(\lambda) := \inf_{\underline{x}_i \le x \le \bar{x}_i} \mathcal{L}_i(x_i, \lambda)$. In particular,

$$J_i(\lambda) = \begin{cases} f_i(\bar{x}_i) + \lambda(\bar{x}_i - p_i^d), & \lambda \le h_i(\bar{x}_i), \\ \frac{(\lambda+b_i)^2}{-4a_i} + c_i - \lambda p_i^d, & h_i(\bar{x}_i) < \lambda < h_i(\underline{x}_i), \\ f_i(\underline{x}_i) + \lambda(\underline{x}_i - p_i^d), & \lambda \ge h_i(\underline{x}_i) \end{cases}$$

where $h_i(x) := -2a_i x - b_i$. (The proof is similar to [11].) Finally, given any optimal solution λ^* to the dual problem (13), we can obtain the unique optimal solution x_i^* to (12) via $x_i^* = \arg\min_{\underline{x}_i \le x_i \le \bar{x}_i} \mathcal{L}_i(x, \lambda^*)$ which can be written as $x_i^* = \min\{\max\{h_i^{-1}(\lambda^*), \underline{x}_i\}, \bar{x}_i\}$ where h_i^{-1} is the inverse function of h_i [10], [12]. Therefore, it is sufficient to solve (13) to obtain the optimal power generation.

In order to solve (13) in a distributed manner, we propose the following algorithm

$$\dot{\lambda}_i = \frac{dJ_i}{d\lambda}(\lambda_i) + k_i(t) \sum_{j\in\mathcal{N}_i} (\lambda_j - \lambda_i), \tag{14a}$$

$$\dot{k}_i = \sum_{j\in\mathcal{N}_i} \sigma_i\left((e_{ij})^2\right) + \sum_{j\in\mathcal{N}_i} (k_j - k_i), \quad \forall i \in \mathcal{N}, \tag{14b}$$

Bus	p_i^d	a_i	b_i	$\bar{x}_i$	Bus	p_i^d	a_i	b_i	$\bar{x}_i$
1	2.17	0.5	0.2	8	8	0	2	0.5	8
2	0	1	0.3	9	9	2.95	0	0	0
3	9.42	1.5	0.4	7	10	0.9	0	0	0
4	4.78	0	0	0	11	0.35	0	0	0
5	0.76	0	0	0	12	0.61	0	0	0
6	1.12	2	0.4	7	13	1.35	0	0	0
7	0	0	0	0	14	1.49	0	0	0

TABLE I: Table shows the parameter used for the simulation.

which has the structure studied in this paper. It can be checked that the system (14) satisfies all assumptions made in previous sections. In particular, $|dJ_i/d\lambda| \leq \max(|\bar{x}_i - p_i^d|, |\underline{x}_i - p_i^d|)$. Hence, the solution of (14) practically converges to the optimal solutions of (13), which is stated below.

Theorem 3: Let Ξ^* be the set of optimal solutions of (13). Suppose that the TUP is used with global performance index $\epsilon > 0$. Then, the solution of (14) practically converges to Ξ^*. In particular, it holds that

$$\limsup_{t\to\infty} |\lambda_i(t)|_{\Xi^*} \leq \epsilon, \quad \forall i \in \mathcal{N}.$$

Proof: Proof is omitted due to space limitation. ■

IV. Simulation

The EDP (12) has been solved using the proposed solution (14) for the IEEE 14 bus system [13] with parameters shown in Table I while $c_i = 0$ and $\underline{x}_i = 0$ were used for all $i \in \mathcal{N}$. Each bus was regarded as a node and it was assumed that two nodes connected by a branch can communicate with each other. We suppose that node 1 through 14 joins the network sequentially in every 50 seconds following the communication network given by [13]. Initial conditions were chosen such that $k_i(0) = 50$, $\lambda_i(0) \in [-200, 200]$ and $\epsilon = 0.1$ was used. Simulation result is shown in Fig. 1 where peaks in every 50 seconds correspond to the addition of a new node. In Fig. 1(a), it is observed that $\lambda_i(t)$ (practically) converges to the optimal solution which is plotted with red dotted line. In Fig. 1(b), it is observed that coupling gains converges to a common, finite value. Moreover, Fig. 1(c) shows that the synchronization error is robust to addition of new agent and stays less than ϵ once converged.

V. Conclusion

A fully distributed algorithm using adaptive law has been presented to achieve practical synchronization of heterogeneous agents with bounded vector fields. The proposed adaptive law adjust the coupling gains based on local information and can be designed and implemented in a fully distributed manner. Moreover, a distributed protocol is proposed to construct a network with guaranteed global performance index regardless of any global information. The proposed solution was applied to the EDP to obtain the optimal power generation in a fully distributed fashion.

Appendix

A. Technical Lemmas

Let us define $x := [x_1^T, \ldots, x_N^T]^T \in \mathbb{R}^{nN}$ and $k := [k_1, \ldots, k_N]^T \in \mathbb{R}^N$. Then, the system (1)-(3) can be written

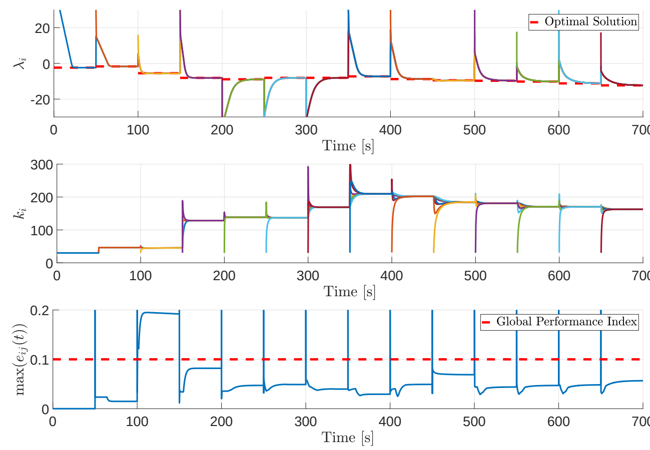

Fig. 1: Graphs show the trajectories of (a) states, (b) coupling gains and (c) synchronization error, i.e., $\max_{i,j\in\mathcal{N}} |e_{ij}(t)|$.

as

$$\dot{x} = F(t, x) - (K(t) \otimes I_n)(L \otimes I_n)x, \tag{15a}$$
$$\dot{k} = G(e) - Lk, \tag{15b}$$

where $e := (B^T \otimes I_n)x \in \mathbb{R}^{En}$, $F(t,x) := [f_1(t,x_1)^T, \ldots, f_N(t,x_N)^T]^T \in \mathbb{R}^{nN}$ and $G(e) := [\sum_{j\in\mathcal{N}_1} \sigma(e_{1j}^T e_{1j}), \ldots, \sum_{j\in\mathcal{N}_N} \sigma(e_{Nj}^T e_{Nj})]^T \in \mathbb{R}^N$. Also let $K(t) := \mathrm{diag}(k_1(t), \ldots, k_N(t)) \in \mathbb{R}^{N\times N}$.

We apply the following transformation,

$$\begin{bmatrix} \bar{\xi} \\ \tilde{\xi} \end{bmatrix} = (W \otimes I_n)x, \quad \begin{bmatrix} \bar{\zeta} \\ \tilde{\zeta} \end{bmatrix} = Wk \tag{16}$$

to (15), where $W \in \mathbb{R}^{N\times N}$ is an invertible matrix given by

$$W = \begin{bmatrix} \frac{1}{N} 1_N^T \\ R^T \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} 1_N & Q \end{bmatrix},$$

such that $WLW^{-1} = \mathrm{diag}(0, \Lambda)$ where $\Lambda := \mathrm{diag}(\mu_2, \ldots, \mu_N) \in \mathbb{R}^{(N-1)\times(N-1)}$ [2]. Moreover, $R, Q \in \mathbb{R}^{N\times(N-1)}$ are matrices satisfying $R^T R = (1/N)I_{N-1}$, $Q^T Q = N I_{N-1}$, $R^T 1_N = 0$, $Q^T 1_N = 0$ and $R^T Q = I_{N-1}$. Then, (15) is transformed into

$$\dot{\bar{\xi}} = \frac{1}{N}(1_N^T \otimes I_n)[F(t,x) - (K(t)LQ \otimes I_n)\tilde{\xi}], \tag{17a}$$
$$\dot{\tilde{\xi}} = (R^T \otimes I_n)F(t,x) - (R^T K(t) LQ \otimes I_n)\tilde{\xi}, \tag{17b}$$
$$\dot{\bar{\zeta}} = \frac{1}{N} 1_N^T G(e), \tag{17c}$$
$$\dot{\tilde{\zeta}} = R^T G(e) - \Lambda\tilde{\zeta}. \tag{17d}$$

Lemma 1: $\tilde{\xi}$ and e are equivalent in terms of their norms. In particular,

$$\alpha_1 |\tilde{\xi}|_\infty \leq |e|_\infty \leq \alpha_2 |\tilde{\xi}|_\infty \tag{18}$$

holds where $\alpha_1 := \sqrt{N\mu_2/En}$ and $\alpha_2 := |B^T|_\infty \cdot |Q|_\infty$.

Proof: By the definition of e, we have

$$|e|_\infty = |(B^T \otimes I_n)x|_\infty = |(B^T \otimes I_n)(Q \otimes I_n)\tilde{\xi}|_\infty$$
$$\leq |B^T|_\infty |Q|_\infty |\tilde{\xi}|_\infty \tag{19}$$

which concludes that the second inequality of (18) holds. On the other hand, note that $|(B^T \otimes I_n)x|_2^2 = |e|_2^2$ holds. Moreover, using $BB^T = L$, we obtain

$$\begin{aligned} x^T(BB^T \otimes I_n)x &= \tilde{\xi}^T(Q^T LQ \otimes I_n)\tilde{\xi} = \tilde{\xi}^T(Q^T Q\Lambda \otimes I)\tilde{\xi} \\ &= N\tilde{\xi}^T(\Lambda \otimes I_n)\tilde{\xi} \geq N\mu_2|\tilde{\xi}|_2^2 \geq N\mu_2|\tilde{\xi}|_\infty^2. \end{aligned}$$

Therefore, using the equivalence of norms, it holds that $N\mu_2|\tilde{\xi}|_\infty^2 \leq |e|_2^2 \leq En|e|_\infty^2$. This concludes that the first inequality of (18) holds. ■

Lemma 2: The coupling gains $k_i(t)$ of the system (1)-(3) is bounded from below. In particular,

$$k_i(t) \geq k_{\min}(t) \geq \underline{k} > 0, \quad \forall t \geq 0, \forall i \in \mathcal{N} \tag{20}$$

where $k_{\min}(t) := \min_{i\in\mathcal{N}} k_i(t)$ and $\underline{k} := k_{\min}(0)$. Moreover, $k_{\min}(t)$ is non-decreasing.

Proof: The first and third inequality of (20) follows directly from the definition. For the second inequality of (20), consider the following scalar differential equation,

$$\dot{z} = 0, \quad z(0) = \underline{k}.$$

Since $k_i(t)$ is continuous for all $i \in \mathcal{N}$, we have

$$\begin{aligned} D_+ k_{\min}(t)^1 &\geq \min_{i\in\mathcal{I}(t)} \Big\{ \sum_{j\in\mathcal{N}_i} \sigma_i(e_{ij}^T e_{ij}) + \sum_{j\in\mathcal{N}_i} (k_j - k_i) \Big\} \\ &\geq 0, \quad \forall t \geq 0 \end{aligned}$$

where $\mathcal{I}(t) := \{i \in \mathcal{N} | k_i(t) = k_{\min}(t)\}$ for all $t \geq 0$. Thus, we conclude $k_{\min}(t)$ is non-decreasing. Also, we obtain $k_{\min}(t) \geq z(t) = \underline{k}$ from the comparison principle [14]. ■

Lemma 3: The following statements hold for the system (1)-(3).

1) $\tilde{\xi}(t)$ and $e(t)$ are bounded.
2) $\tilde{\zeta}(t)$ is bounded.

Proof: In order to show the boundedness of $\tilde{\xi}(t)$, let $V(\tilde{\xi}) = \frac{1}{2}\tilde{\xi}^T(U \otimes I_n)\tilde{\xi}$, where $U := NR^T LQ$. From the properties of the transformation matrix W, it can be shown that $U = U^T > 0$ and $RU = LQ$. Then, the time derivative of V becomes

$$\begin{aligned} \dot{V} &= \tilde{\xi}^T(U \otimes I)\left[(R^T \otimes I)F(t, x) - (R^T K(t)LQ \otimes I)\tilde{\xi}\right] \\ &\leq |U||R||F(t, x)||\tilde{\xi}|_2 - \tilde{\xi}^T(Q^T L^T K(t)LQ \otimes I_n)\tilde{\xi} \\ &\leq (\mu_N N)(\frac{1}{\sqrt{N}})M|\tilde{\xi}|_2 - k_{\min}(t)N\mu_2^2|\tilde{\xi}|_2^2 \end{aligned}$$

where $M > 0$ is a constant such that $|F(t, x)| < M$ whose existence is assured by Assumption 1. This concludes that $\tilde{\xi}(t)$ is (ultimately) bounded since $k_{\min}(t)$ is non-decreasing by Lemma 2. In particular, for any fixed time $t^* > 0$, there exists $T' > 0$ such that $\tilde{\xi}(t)$ satisfies

$$|\tilde{\xi}(t)|_2 \leq \frac{2\mu_N^2 M}{k_{\min}(t^*)\sqrt{N}\mu_2^3}, \quad \forall t \geq T'. \tag{21}$$

Consequently, the boundedness of $e(t)$ follows directly from Lemma 1.

For the boundedness of $\tilde{\zeta}$, let $V(\tilde{\zeta}) = \frac{1}{2}\tilde{\zeta}^T\tilde{\zeta}$. Then,

$$\begin{aligned} \dot{V} = \tilde{\zeta}^T\dot{\tilde{\zeta}} = \tilde{\zeta}^T\left[R^T G(e) - \Lambda\tilde{\zeta}\right] &\leq |\tilde{\zeta}||R^T||G(e)| - \mu_2|\tilde{\zeta}|^2 \\ &\leq \frac{c_1}{\sqrt{N}}|\tilde{\zeta}| - \mu_2|\tilde{\zeta}|^2 \end{aligned}$$

where $c_1 > 0$ is a constant such that $|G(e)| \leq c_1$, which exists since e is bounded and $G(e)$ is continuous. Therefore, this concludes that $\tilde{\zeta}(t)$ is bounded. ■

Lemma 4: The following statements hold for the system (1)-(3).

1) $k_i(t)$ is bounded for all $i \in \mathcal{N}$.
2) $\bar{\zeta}(t)$ is bounded.
3) $k_{\min}(t)$ is bounded.

Proof: Suppose that $k_i(t)$ is bounded for all $i \in \mathcal{N}$. Then, it is easy to see that $\bar{\zeta}(t) = (1/N)\sum k_i(t)$ is bounded. Moreover, the boundedness of $k_{\min}(t)$ also follows from (20). Therefore, it is sufficient to show the boundedness of the coupling gains $k_i(t)$.

To use proof by contradiction, suppose there exists an index $i \in \mathcal{N}$ such that $k_i(t)$ is not bounded. Then, for all $B_1 > 0$, there exists $T_1 > 0$ such that $|k_i(T_1)| > B_1$. However, recalling that $k = 1_N\bar{\zeta} + Q\tilde{\zeta}$ holds from the inverse transformation W^{-1}, we obtain

$$|k_i(t)| = |\bar{\zeta}(t) + (Q\tilde{\zeta}(t))_i| \leq |\bar{\zeta}(t)| + |Q\tilde{\zeta}(t)|, \quad \forall t \geq 0$$

where $(Q\tilde{\zeta}(t))_i$ denotes the i^{th} component of the vector $Q\tilde{\zeta}(t)$. This leads to

$$B_1 - |Q\tilde{\zeta}(T_1)| \leq |\bar{\zeta}(T_1)|.$$

Thus, $\bar{\zeta}(t)$ is unbounded since B_1 can be arbitrarily large and $\tilde{\zeta}(t)$ is bounded by Lemma 3. In turn, $k_i(t)$ is unbounded for all $i \in \mathcal{N}$ since $k = 1_N\bar{\zeta} + Q\tilde{\zeta}$.

Hereafter, we show $k_{\min}(t)$ is also unbounded. Suppose that $k_{\min}(t)$ is bounded, i.e., there exists $B_2 > 0$ such that $|k_{\min}(t)| < B_2$ for all $t \geq 0$. Since $\bar{\zeta}$ is unbounded, there exists T_2 such that $\bar{\zeta}(T_2) > B_2 + c_2$, where $c_2 > 0$ is constant satisfying $|(Q\tilde{\zeta})| \leq c_2$. Since $k_{\min}(t)$ is bounded, there exists at least one index $l \in \mathcal{N}$ such that $|k_l(T_2)| < B_2$. However,

$$\begin{aligned} |k_l(T_2)| &= |\bar{\zeta}(T_2) - (-Q\tilde{\zeta}(T_2))_l| \\ &\geq \Big||\bar{\zeta}(T_2)| - |(-Q\tilde{\zeta}(T_2))_l|\Big| \geq \bar{\zeta}(T_2) - c_2 > B_2 \end{aligned}$$

which leads to contradiction since l was chosen such that $|k_l(T_2)| < B_2$ holds. Therefore, $k_{\min}(t)$ is unbounded.

Since $k_{\min}(t)$ is unbounded and non-decreasing, for any $B_3 > 0$, there exists $T_3 > 0$ such that $k_{\min}(t) > B_3$ for all $t \geq T_3$. Thus, there exists $T_4 > T_3$ such that $\tilde{\xi}(t)$ satisfies

$$|\tilde{\xi}(t)|_2 \leq \frac{2\mu_N^2 M}{B_3\sqrt{N}\mu_2^3}, \quad \forall t \geq T_4$$

by (21). Hence, it follows from Lemma 1 that B_3 can be chosen sufficiently large such that

$$e_{ij}^T(t)e_{ij}(t) < \min(\gamma_i, \gamma_j), \quad \forall (i, j) \in \mathcal{E}, \ \forall t \geq T_4 \tag{22}$$

[1] $D_+v(t)$, the Dini's derivative of $v(t)$, is defined as $\liminf_{h\to 0^+} \frac{v(t+h)-v(t)}{h}$.

is satisfied. If (22) is satisfied, $\sigma_i(e_{ij}^T e_{ij}) = 0$ for all $(i,j) \in \mathcal{E}$ and for all $t \geq T_4$. Thus, for all $t \geq T_4$, (3) reduces to $\dot{k} = -Lk$ which is exactly the classical average consensus [15] where its solutions converge to the average of initial conditions. In particular, $\lim_{t\to\infty} k_i(t) = (1/N)\sum k_i(T_4)$ which implies that the coupling gains are bounded. This leads to contradiction since we assumed there exists at least one coupling gain which is unbounded. Thus, we conclude that coupling gains $k_i(t)$ are bounded. ∎

The next lemma shows that $\dot{e}(t)$ is also bounded by using the boundedness of $k_i(t)$.

Lemma 5: $\dot{e}(t)$ of the system (1)-(3) is bounded.

Proof: From the definition of $e(t)$, its derivative can be written as

$$|\dot{e}| = |(B^T \otimes I_n)\dot{x}| = |(B^T \otimes I)(F(t,x) - (K(t)L \otimes I)x)| \\ \leq |B^T||F(t,x)| + |B^T||K(t)||Q||\tilde{\xi}| \leq c_3$$

where the existence of $c_3 > 0$ follows from the fact that $F(t,x)$, $K(t)$ and $\tilde{\xi}(t)$ are all bounded. ∎

B. Proof of Theorem 1

Now we provide the proof of Theorem 1. First, we show that (4) holds by contradiction. Suppose that there exist an edge $(i,j) \in \mathcal{E}$ such that

$$\limsup_{t\to\infty} e_{ij}^T(t)e_{ij}(t) > \gamma_i. \quad (23)$$

From the definition of limit superior, (23) implies there exists $\delta > 0$ such that for all $T > 0$, there exists $t^* \geq T$ satisfying

$$e_{ij}(t^*)^T e_{ij}(t^*) \geq \gamma_i + \delta.$$

Also note that there exists some constant $c > 0$ such that

$$\left|\frac{d}{dt}(e_{ij}^T e_{ij})\right| = \left|\dot{e}_{ij}^T e_{ij} + e_{ij}^T \dot{e}_{ij}\right| \leq c, \quad \forall t \geq 0$$

by Lemmas 3 and 5. Now, we claim that

$$\lim_{t\to\infty} \int_0^t \sigma_i(e_{ij}^T(\tau)e_{ij}(\tau))d\tau = +\infty. \quad (24)$$

To prove the claim, first let $\{t_k\}_{k=1,\ldots,\infty}$ be the sequence of time such that $t_{k+1} - t_k \geq 2\delta/c$ and $e_{ij}(t_k)^T e_{ij}(t_k) \geq \gamma_i + \delta$. Then, according to the definition of the deadzone function,

$$\sigma_i\left(e_{ij}^T(t_k)e_{ij}(t_k)\right) \geq (\gamma_i + \delta) - \gamma_i = \delta.$$

Therefore, from the continuity of $e_{ij}(t)$ and the boundedness of the derivative of $e_{ij}^T(t)e_{ij}(t)$, we have

$$\lim_{t\to\infty} \int_0^t \sigma_i(e_{ij}^T(\tau)e_{ij}(\tau))d\tau \geq \sum_{k=1}^{\infty} \frac{\delta^2}{c}. \quad (25)$$

Since the right-hand side of (25) diverges, the claim is proved.

Meanwhile, it can be seen from (17c) that

$$\bar{\zeta}(t) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\int_0^t \sigma_i(e_{ij}^T(\tau)e_{ij}(\tau))d\tau + \bar{\zeta}(0). \quad (26)$$

However, (24) with (26) implies that $\bar{\zeta}(t)$ diverges, which contradicts with Lemma 4. This completes the proof of (4).

In order to prove the convergence of coupling gains $k_i(t)$, it suffice to show that $\bar{\zeta}(t)$ converges to a finite value and $\tilde{\zeta}(t)$ converges to 0 since $k(t) = 1_N\bar{\zeta}(t) + Q\tilde{\zeta}(t)$. It is easy to see that $\dot{\bar{\zeta}}(t) \geq 0$ holds from (17c). Since $\bar{\zeta}(t)$ is already shown to be bounded in Lemma 4, $\bar{\zeta}(t)$ is non-decreasing and bounded. Thus, it converges to a finite value.

Since $\limsup_{t\to\infty} e_{ij}^T(t)e_{ij}(t) \leq \gamma_i$ for all $(i,j) \in \mathcal{E}$ by (4), it follows that $\lim_{t\to\infty} \sigma_i(e_{ij}^T(t)e_{ij}(t)) = 0$ holds for all $(i,j) \in \mathcal{E}$ because $\sigma_i(e_{ij}^T(t)e_{ij}(t)) \geq 0$. Consequently, this implies that $\lim_{t\to\infty} R^T G(e(t)) = 0$. Thus, it follows from (17d) that $\lim_{t\to\infty} \tilde{\zeta}(t) = 0$ holds since $-\Lambda$ is a Hurwitz matrix. Therefore, coupling gains $k_i(t)$ converge to a finite value. ∎

References

[1] J. M. Montenbruck, M. Bürger, and F. Allgöwer, "Practical synchronization with diffusive couplings," *Automatica*, vol. 53, pp. 235–243, 2015.

[2] J. Kim, J. Yang, H. Shim, J. S. Kim, and J. Seo, "Robustness of synchronization of heterogeneous agents by strong couping and a large number of agents," *IEEE Trans. Autom. Control*, vol. 60, no. 10, pp. 3096–3102, 2016.

[3] E. Panteley and A. Loría, "Synchronization and dynamic consensus of heterogeneous networked systems," *IEEE Trans. Autom. Control*, vol. 62, no. 8, pp. 3758–3773, 2017.

[4] H. Shim and S. Trenn, "A preliminary result on synchronization of heterogeneous agents via funnel control," in *Proc. IEEE Conf. Dec. Control*, pp. 2229–2234, 2015.

[5] S. Shafi and M. Arcak, "An adaptive algorithm for synchronization in diffusively-coupled systems," in *Proc. Amer. Control Conf.*, pp. 2220–2225, 2014.

[6] Z. Li, W. Ren, X. Liu, and M. Fu, "Consensus of multi-agent systems with general linear and Lipschitz nonlinear dynamics using distributed adaptive protocols," *IEEE Trans. Autom. Control*, vol. 58, no. 7, pp. 1786–1791, 2013.

[7] Y. Lv, Z. Li, and Z. Duan, "Novel distributed robust adaptive consensus protocols for linear multi-agent systems with directed graphs and external disturbances," *Int. J. Control*, vol. 90, no. 2, pp. 137–147, 2017.

[8] H. Kim and C. De Persis, "Adaptation and disturbance rejection for output synchronization of incrementally output-feedback passive systems," *Int. J. Robust Nonlin. Control*, vol. 27, no. 17, pp. 4071–4088, 2017.

[9] S. Yang, S. Tan, and J. Xu, "Consensus based approach for economic dispatch problem in a smart grid," *IEEE Trans. Power Syst.*, vol. 28, no. 4, pp. 4416–4426, 2013.

[10] S. Boyd, and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

[11] M. Madrigal and V. H. Quintana, "An analytical solution to the economic dispatch problem," *IEEE Power Eng. Rev.*, vol. 20, no. 9, pp. 52–55, 2000.

[12] T. Yang *et al.*, "A distributed algorithm for economic dispatch over time-varying directed networks with delays," *IEEE Trans. Ind. Electron.*, vol. 64, no. 6, pp. 5095–5106, 2017.

[13] P. Iyambo and R. Tzoneva, "Transient stability analysis of the IEEE 14-bus electric power system," in *AFRICON*, Windhoek, pp. 1–9, 2007.

[14] H. K. Khalil, *Nonlinear Systems,* 3rd ed. Prentice Hall, 2002.

[15] W. Ren, R. Beard, and E. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71–82, 2007.